FrB09 Regular Session, Oceania VIII: **Estimation I**

Parameter Robustness in Estimation of Dynamical Systems

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Conference on Decision and Control CDC2025 Rio de Janeiro, Brazil. Dec. 09 - 12

How to estimate robustness of estimation error under parametric uncertainties?

System Dynamics

$$\dot{x} = A(\theta)x + B(\theta)u,$$
$$y = C(\theta)x + D(\theta)u$$

Estimated Dynamics

$$\tilde{y} = \tilde{C}(\theta)\tilde{x} + \tilde{D}(\theta)u\dot{\tilde{x}} = \tilde{A}(\theta)\tilde{x} + \tilde{B}(\theta)u,$$

Problem:

How does estimation error change as parameters vary?

For perturbations in θ, compute how the estimation error changes:

$$||y - \tilde{y}||$$

An alternative perspective on parametric robustness

Sensitivity-based metric captures the effect of individual parameters

We propose a robustness distance metric using sensitivity of the error with parameters,

$$S_e = \left\| \frac{\partial \text{error}}{\partial \theta} \right\|$$

Then, we define a robustness distance (for all p parameters) by normalizing:

$$d_R = \sum_{i=1}^{p} \frac{\theta^*}{\|\text{error}\ (t,\ \theta^*)\|} S_e$$

And a robustness metric R:

$$R = 1/(1 + d_R)$$

But computing sensitivities for all parameters for all time is inefficient...

System Dynamics

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Estimated Dynamics

$$\tilde{y} = \tilde{C}(\theta)\tilde{x} + \tilde{D}(\theta)u\dot{\tilde{x}} = \tilde{A}(\theta)\tilde{x} + \tilde{B}(\theta)u,$$

To compute:

$$\frac{\partial\,(y - \tilde{y})}{\partial\theta}$$

we need to solve the sensitivity ODE (a 2n size ODE) for both systems!

So, we look for a computationally efficient bound.

Augmented system dynamics naturally offers the estimation error as an output

System Dynamics

$$\dot{x} = A(\theta)x + B(\theta)u,$$
$$y = C(\theta)x + D(\theta)u$$

Estimated Dynamics

$$\tilde{y} = \tilde{C}(\theta)\tilde{x} + \widetilde{D}(\theta)u\dot{\tilde{x}} = \tilde{A}(\theta)\tilde{x} + \widetilde{B}(\theta)u,$$

$$\bar{A} = \begin{bmatrix} \mathrm{A} & 0 \\ 0 & \tilde{A} \end{bmatrix}, B = \begin{bmatrix} \mathrm{B} \\ \tilde{B} \end{bmatrix}, C = \begin{bmatrix} C & -\tilde{C} \end{bmatrix}, D = D - \widetilde{D}$$

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u$$
$$\bar{y} = \bar{C}\bar{x} + \bar{D}u$$

Estimation error

But computing sensitivities for all parameters for all time is inefficient...

Augmented (Error) Dynamics

$$\dot{\bar{x}} = \bar{\mathrm{A}}(\theta)\bar{x} + \bar{B}(\theta)\mathrm{u}$$

$$\bar{y} = \bar{C}(\theta)\bar{x} + \bar{D}(\theta)u$$

To compute:

$$\frac{\partial \bar{y}}{\partial \theta}$$

we need to solve the sensitivity ODE (a 2n size ODE) for both systems just for one parameter!

So, we look for a computationally efficient bound... $\bar{x}(t) = e^{\bar{\mathrm{A}}t}\bar{x}(0), \qquad \mathrm{y} = \bar{C}e^{\bar{\mathrm{A}}t}\bar{x}(0)$

$$\frac{\partial \bar{x}}{\partial \theta} = e^{\bar{\mathrm{A}}t}\frac{\partial \bar{x}(0)}{\partial \theta} + \frac{\partial e^{\bar{\mathrm{A}}t}}{\partial \theta}\bar{x}(0)$$

How can we compute the perturbed matrix exponential?

Augmented (Error) Dynamics

$$\dot{\bar{x}} = \bar{\mathrm{A}}(\theta)\bar{x} + \bar{B}(\theta)\mathrm{u}$$

$$\bar{y} = \bar{C}(\theta)\bar{x} + \bar{D}(\theta)u$$

To compute:

$$\frac{\partial\,(\mathrm{y} - \tilde{\mathrm{y}})}{\partial\theta}$$

we need to solve the sensitivity ODE (a 2n size ODE) for both systems!

So, we look for a computationally efficient bound... $\bar{x}(t) = e^{\bar{\mathrm{A}}t}\bar{x}(0)$, $\quad \mathrm{y} = \bar{C}e^{\bar{\mathrm{A}}t}\bar{x}(0)$

$$\frac{\partial\bar{x}}{\partial\theta} = e^{\bar{\mathrm{A}}t}\frac{\partial\bar{x}(0)}{\partial\theta} + \frac{\partial e^{\bar{\mathrm{A}}t}}{\partial\theta}\bar{x}(0)$$

The robustness bound depends on the derivative of the matrix exponential, which in turn comes from a result on "perturbation variation" by Snider 1964*

Proof from Snider 1964

$$\frac{\partial}{\partial \lambda} e^{z(\lambda)} = \sum_{n=0}^{\infty} \sum_{m=0}^{n-1} \frac{1}{n!} z^m \frac{\partial z}{\partial \lambda} z^{n-m-1},$$

$$= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \frac{m!p!}{(m+p+1)!} \frac{z^m}{m!} \frac{\partial z}{\partial \lambda} \frac{z^p}{p!},$$

$$= \sum_{m,p} \int_0^1 (1-\alpha)^m \alpha^p \, d\alpha \frac{z^m}{m!} \frac{\partial z}{\partial \lambda} \frac{z^p}{p!},$$

$$= \int_0^1 e^{(1-\alpha)z} \frac{\partial z}{\partial \lambda} e^{\alpha z} \, d\alpha,$$

Alternatively...

With,

$$S(t) = \frac{\partial x(t)}{\partial \theta}$$

$$\dot{S} = AS + \frac{\partial A}{\partial \theta} x \quad \text{(sensitivity equation)}$$

$$S(t) = e^{At} S(0) + x(0) \int_0^t e^{A(t-\tau)} \frac{\partial A}{\partial \theta} e^{A\tau} d\tau$$

Bound on derivative of matrix exp

$$\left\| \frac{\partial e^{At}}{\partial \theta} \right\| \leq \left\| \frac{\partial A}{\partial \theta} \right\| t e^{-\mu t}$$

*also appears in (Wilcox 1967) and (Tsai and Chan 2003)

Now we can compute a bound on the error sensitivity using the perturbed matrix exp

Augmented (Error) Dynamics

$$\dot{\bar{x}} = \bar{\mathrm{A}}(\theta)\bar{x} + \bar{B}(\theta)\mathrm{u}$$

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To compute:

$$\frac{\partial\,(\mathrm{y} - \tilde{\mathrm{y}})}{\partial\theta}$$

we need to solve the sensitivity ODE (a 2n size ODE) for both systems!

So, we look for a computationally efficient bound…

$$\bar{x}(t) = e^{\bar{\mathrm{A}}t}\bar{x}(0), \qquad \mathrm{y} = \bar{C}e^{\bar{\mathrm{A}}t}\bar{x}(0)$$

$$\frac{\partial\bar{x}}{\partial\theta} = e^{\bar{\mathrm{A}}t}\frac{\partial\bar{x}(0)}{\partial\theta} + \frac{\partial e^{\bar{\mathrm{A}}t}}{\partial\theta}\bar{x}(0)$$

Robustness of estimation error when only the matrix A is parameter dependent

Theorem 1

A simple derivative of system matrix

Total time for which inputs are applied

$$\left\|\frac{\partial \bar{y}}{\partial \theta_i}\right\|^2 \leq K_1 \left\|\frac{\partial \bar{A}}{\partial \theta_i}\right\|^2 + K_2 \left\|\frac{\partial \bar{A}}{\partial \theta_i}\right\|^3 \|\bar{B}u\|_\infty + K_3 N^2 \left\|\frac{\partial \bar{A}}{\partial \theta_i}\right\|^2 \|\bar{B}u\|_\infty$$

Norm of the estimation error

Worst-case norm on inputs

Robustness of estimation error when only initial conditions are parameter dependent

Theorem 2

A simple derivative of initial conditions

$$\left\| \frac{\partial \bar{y}}{\partial \theta_i} \right\|^2 \leq \lambda_{\max}(P) \left\| \frac{\partial \bar{x}(0)}{\partial \theta_i} \right\|^2$$

Norm of the estimation error

Max eigen value of Lyapunov matrix

Example Setting

System Dynamics

$$A(\theta) = \begin{bmatrix} 0 & 1 \\ -(20 + 5\,\theta) & -(2 + 0.5\,\theta) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0], \quad D = 0$$

Estimated Dynamics

$$\tilde{A}(\theta) = \begin{bmatrix} 0 & 1 \\ -(19.8 + 5.1\,\theta) & -(2.05 + 0.48\,\theta) \end{bmatrix}$$

$$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0], \quad D = 0$$

Goal:

As parameter(s) vary,

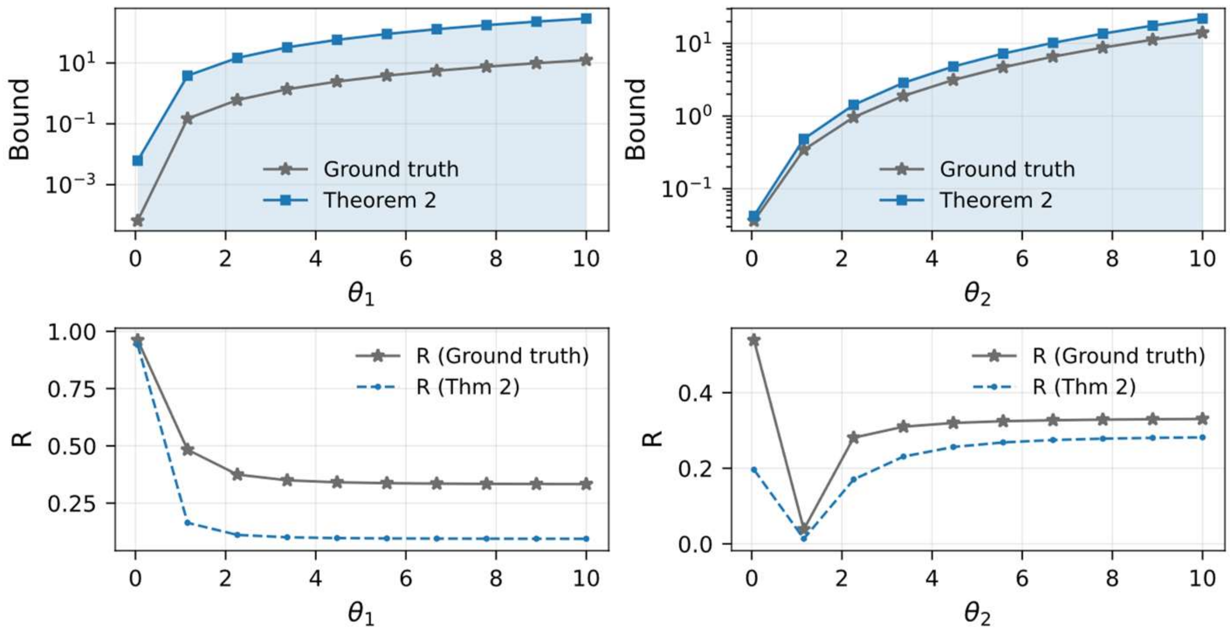
Bound $||\partial \bar{y}/\partial \theta||$

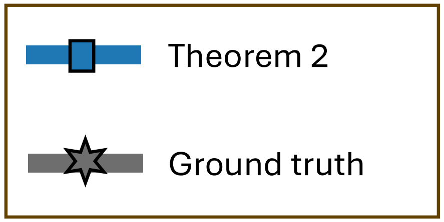
Compute R (robustness metric)

Notes:

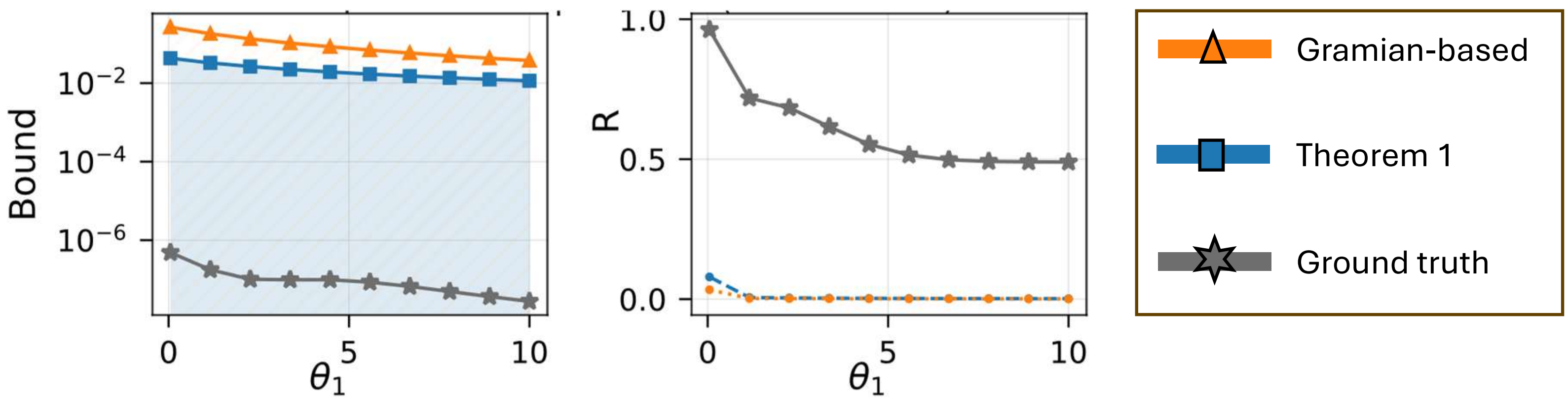
- Linear dependence on parameters is not required.
- Robustness to individual parameters can be isolated
- Estimated A must be representable in the same structure as original A

The sensitivity-based robustness computation is tight when initial conditions are parametric

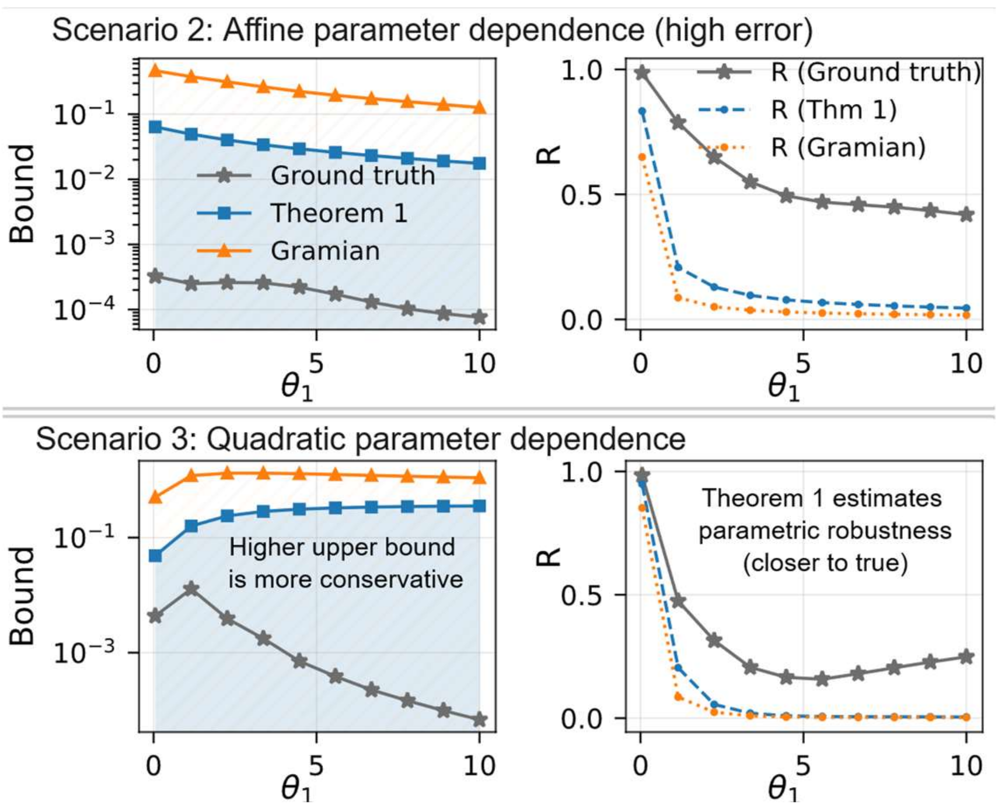


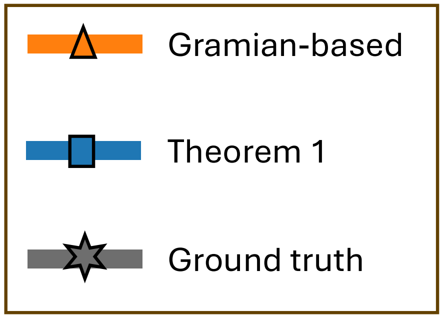


The sensitivity-based robustness computation can be very conservative if error is minimal / parameter dependence is low



Sensitivity-based robustness computation is useful when parameter dependence is significant





Take-home messages

1. Parametric robustness of estimation error can be quantified using sensitivity-based bounds that are easily computable (do not require you to solve an ODE)....if the estimated system can be represented similar to the original system.

2. The sensitivity-based bounds can isolate and quantify the dependence on individual parameters so you can design estimators that accept fragility to certain parameters while ensuring robustness to others.

3. Future work: Neural networks as estimators can naturally fit this system setting. Parameter perturbation analysis can be a useful step towards interpretability.

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Apply the method to other system examples: `robest` on GitHub

ayush9pandey/robest

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